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Some graphs arising from derivations

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Abstract

Let R be a ring and d a derivation of R. The graph $\Gamma_d(R)$ of R is defined as the graph with vertices set R, and two distinct vertices x and y are adjacent if and only if $[d(x), d(y)] + d\,[x, y] - [x, y] = 0$. In this paper, we study the interaction between some ring-theoretic properties of R and certain graph-theoretic properties of $\Gamma_d(R)$.

Key words: Prime ring, derivations, commutativity, complete graph, vertices, edges, connected graph, star graph, centerlike.

2020 Mathematics Subject Classification: 05C25, 16N60, 16W25.

1. Introduction

In the past thirty years, the study of algebraic structures via properties of graphs has developed into an interesting research topic, producing different fascinating results and challenges. The study of graph theoretical aspects of rings has been initiated by Beck in 1988 [4], where he introduced the concept of a zero-divisor graph of a commutative ring R with identity. Furthermore, various graphs can be associated to rings or other algebraic structures which allows us to discover more about algebraic structures and vice versa.

Over the last years, a number of researchers have investigated the connection between commutativity, the structure of a ring R and the behaviour of certain additive mappings of R. In this direction, Posner [10] proved that a prime ring R must be commutative if it admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x, y \in R$ then R is commutative. Since then, several authors have extended Posner's result in various directions.

Our principal aim in this paper is to introduce a new class of graph for a differential ring (R, d). In fact, to each derivation d of a ring R we associate a graph $\Gamma_d(R)$ where the adjacency relation is defined by some algebraic identity involving d.

2. Background

Throughout this paper, R denotes a ring, with center Z. Recall that a ring R is said to be a 2-torsion free if whenever $2x = 0$ implies that $x = 0$. A ring R is called a prime ring if $xRy = \{0\}$, with $x, y \in R$, implies that $x = 0$ or $y = 0$. An additive mapping $d : R \longrightarrow R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A mapping $f : R \longrightarrow R$ is called strong commutativity preserving (SCP) if $[x, y] = [f(x), f(y)]$ for all $x, y \in R$. The Lie product is defined by $[x, y] = xy - yx$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

For a graph Γ, $V(\Gamma)$ stands for the set of vertices and the set of edges is denoted $E(\Gamma)$, the edges will be represented by drawing a segment between two adjacent vertices (x)—(y).

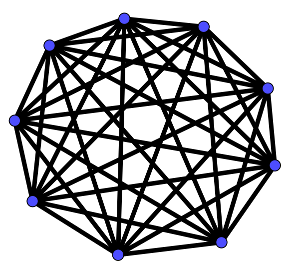
A graph is said to be connected if every pair of vertices in the graph is connected, which means that there is a path between every pair of vertices. A graph Γ is complete if the set of edges $E(\Gamma) = V(\Gamma) \times V(\Gamma) \setminus \{(x, x) | x \in V(\Gamma)\}$. Star graph is a form of graph that has $n - 1$ vertices with degree 1 and a single vertex with degree $n - 1$. A cycle exists when the first and last vertices of a chain are the same. The length of the shortest cycle in Γ, given by $gr(\Gamma)$, is the girth of Γ. The distance between two vertices x and y is the length of the shortest path between them, we note it by $dist(x, y)$. The diameter of Γ is defined as

$$diam(\Gamma) = \max_{x,y \in V(\Gamma)} dist(x, y).$$

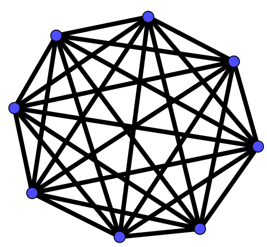
For a ring R, the graph denoted by $\Gamma_d(R)$, is a simple undirected graph with $V(\Gamma_d(R)) = R$, and two distinct vertices x and y are adjacent if and only if $[d(x), d(y)] + d\,[x, y] - [x, y] = 0$. By $\Gamma_d(R^*)$ we mean the graph of R such that $V(\Gamma_d(R^*)) = V(\Gamma_d(R)) \setminus \{0\}$. The following figures represent different examples of $\Gamma_d(R)$ and $\Gamma_d(R^*)$:

Examples.

1) Let us consider $R_1 = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}/3\mathbb{Z} \right\}$ and d_1 a derivation of R, such that $d_1 \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we get $\Gamma_{d_1}(R_1)$ and $\Gamma_{d_1}(R_1^*)$ as follows



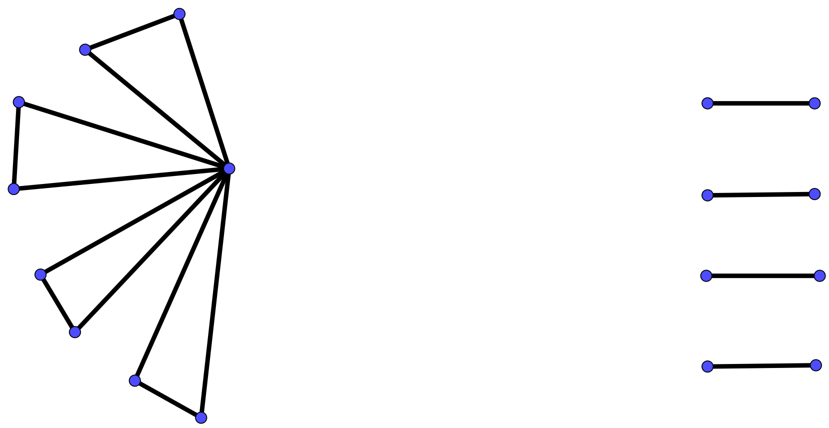
(a) $\Gamma_d(R_1)$



(b) $\Gamma_d(R_1^*)$

Figure 1

We continue to use the previous ring R_1, but we consider d_2 as a new derivation of R_1, such that $d_2 \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b-a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we get



(a) $\Gamma_{d_2}(R_1)$ (b) $\Gamma_{d_2}(R_1^*)$

Figure 2

In light of figures 1 and 2, we can see that a simple modification in the derivation leads to the distinct graphs : $\Gamma_{d_1}(R_1)$ and $\Gamma_{d_2}(R_1)$, that on the one hand, on the other if we keep using the same derivation but remove the vertex (0), the connected graph $\Gamma_{d_2}(R_1)$ becomes the non-connected graph $\Gamma_{d_2}({R_1}^*)$, but this is not the case for the two graphs $\Gamma_{d_1}(R_1)$ and $\Gamma_{d_1}({R_1}^*)$ due to the completeness property.

2) Consider $R_2 = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, f \in \mathbb{Z}/2\mathbb{Z} \right\}$ and h_1 a derivation of R, such that $h_1 \begin{pmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b+a & c+a+f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we obtain $\Gamma_{h_1}(R_2)$ and $\Gamma_{h_1}({R_2}^*)$ as follows

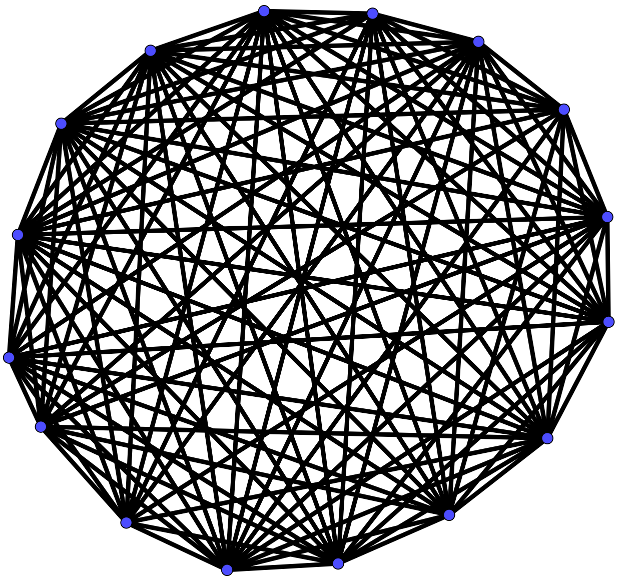


Figure 3: The graph $\Gamma_{h_1}(R_2)$.

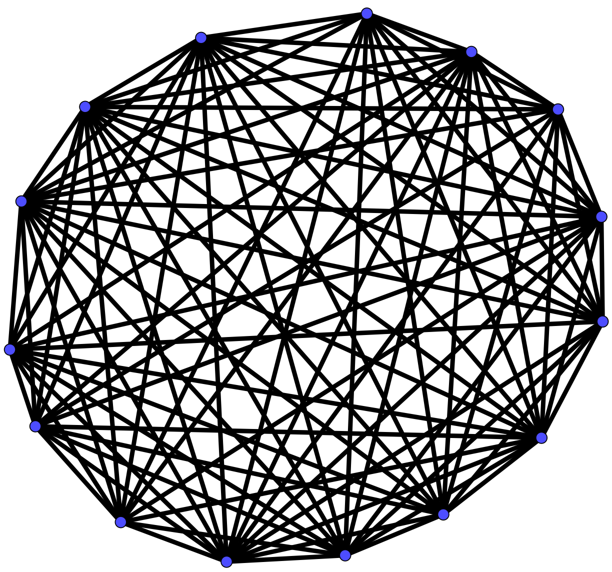
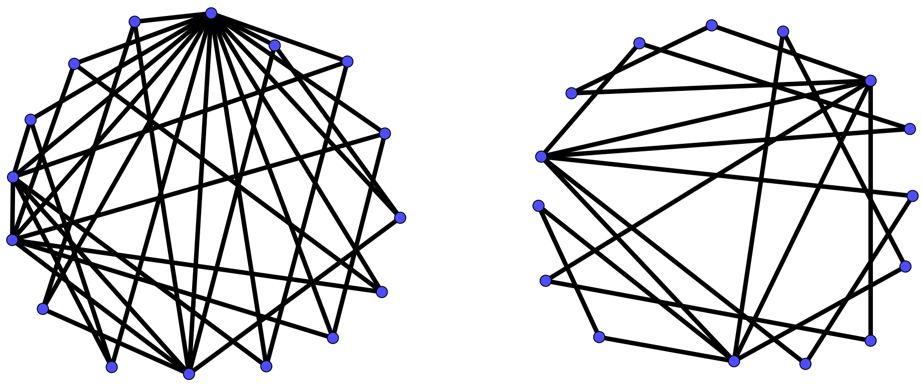


Figure 4: The graph $\Gamma_{h_1}({R_2}^{*})$.

We are still using the previous ring R_2, but this time we consider h_2 as another derivation of R_2, so that $h_2\begin{pmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & a+b+f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we find that $\Gamma_{h_2}(R_2)$ and $\Gamma_{h_2}(R_2{}^*)$ as follows



(a) $\Gamma_{h_2}(R_2)$

(b) $\Gamma_{d_2}(R_2{}^*)$

Figure 5

According to figures 3, 4 and 5, we can see that a simple modification in the derivation leads to the distinct graphs : $\Gamma_{h_1}(R_2)$ and $\Gamma_{h_2}(R_2)$. Additionally, we see that the completeness of $\Gamma_{h_1}(R_2{}^*)$ is unaffected by the modification in the set of vertices, the same for the connectivity of $\Gamma_{h_2}(R_2{}^*)$.

3. Main results

The following lemma is very crucial for developing the proofs of our main results.

Lemma 1. *([9], Lemma 3.10). Let R be a prime ring of characteristic not 2 and suppose $a, b \in R$ are such that $arb + bra = 0$ for all $r \in R$. Then either a=0 or $b = 0$.*

Now, we investigate the diameter of $\Gamma_d(R^*)$ and $\Gamma_d(R)$.

Proposition 1. *Let R be a ring and d a derivation of R. If $Z(R)^*$ contains at least one element then $Diam(\Gamma_d(R^*)) \leq 2$. Furthermore, if $\Gamma_d(R^*)$ is not a complete graph then $Diam(\Gamma_d(R^*)) = 2$.*

Proof. Let $z \in Z(R)^*$, thus $[d(z), d(y)] + d[z, y] - [z, y] = 0$ for all $y \in R$, then there is an edge from vertex (z) to vertex (y) for all $y \in V(\Gamma_d(R^*))$ (z)—(y), which means that any two vertices of $\Gamma_d(R^*)$ are connected by the vertex (z), (x)—(z)—(y). Hence $\Gamma_d(R^*)$ is a connected graph. Since the vertex (z) connects all the others vertices by a chain then $dist(x, y) \leq 2$ for any two vertices $x, y \in V(\Gamma_d(R^*))$. Thus $Diam(\Gamma_d(R^*)) \leq 2$. If $\Gamma_d(R^*)$ is a complete graph, then we get $dist(x, y) = 1$ for all $x, y \in R$, however if $\Gamma_d(R^*)$ is not a complete graph, we obtain $Diam(\Gamma_d(R^*)) = 2$. ∎

The following corollary is an immediate consequence of the above proposition.

Corollary 1. *Let R be a ring and d a derivation of R. $\Gamma_d(R)$ is a connected graph and $Diam(\Gamma_d(R)) \leq 2$. Furthermore, if $\Gamma_d(R)$ is not a complete graph then $Diam(\Gamma_d(R)) = 2$.*

The next corollaries conclude the girth of $\Gamma_d(R^*)$ and $\Gamma_d(R)$.

Corollary 2. *Let R be a ring, d a derivation of R and $|Z(R)^*| \geq 1$. If $\Gamma_d(R^*)$ contains a cycle, then $gr(\Gamma_d(R^*)) = 3$.*

Proof. By assumption $\Gamma_d(R^*)$ is a connected graph, if $\Gamma_d(R^*)$ contains a cycle then the cycle will be presented in that form (z)—(x)—(y)—(z), then $gr(\Gamma_d(R^*)) = 3$. ∎

Corollary 3. *Let R be a ring and d a derivation of R. If $\Gamma_d(R)$ contains a cycle, then $gr(\Gamma_d(R)) = 3$.*

Motivated by the results of [5], we prove the following Theorem.

Theorem 1. *Let R be a 2-torsion free prime ring, d a derivation of R and y an element of R, then y is adjacent to x for all $x \in R$ if and only if $y \in Z(R)$.*

Proof. Assume that y is adjacent to x for all $x \in R$. Then we have

$$[d(x), d(y)] + d[x, y] - [x, y] = 0 \quad \text{for all} \quad x \in R. \tag{1}$$

Substituting xy for x in (1) and applying it, we obtain

$$d(x)[y, d(y)] + [x, d(y)]d(y) + [x, y]d(y) = 0 \quad \text{for all } x \in R. \tag{2}$$

Writing xr instead of x in (2), we obviously get

$$d(x)r[y, d(y)] + [x, d(y)]rd(y) + [x, y]rd(y) = 0 \quad \text{for all } x, r \in R. \tag{3}$$

Putting $d(y) = x$ in (3), we arrive at

$$d^2(y)r[y, d(y)] + [d(y), y]rd(y) = 0 \quad \text{for all } r \in R. \tag{4}$$

Replacing x by y in (3), we obtain

$$d(y)r[y, d(y)] + [y, d(y)]rd(y) = 0 \quad \text{for all } r \in R. \tag{5}$$

Adding (4) together with (5), one can see that

$$\big(d^2(y) + d(y)\big)r[y, d(y)] = 0 \quad \text{for all } r \in R. \tag{6}$$

Invoking the primeness of R, the last relation assures that $d^2(y)+d(y) = 0$ or $[y, d(y)] = 0$. Now if $d^2(y) + d(y) = 0$, then $[d^2(y), d(y)] = 0$ and by taking $x = d(y)$ in the relation (1), we get

$$2[y, d(y)] = 0.$$

Using 2-torsion freeness, it follows that $[y, d(y)] = 0$. Whence in any cases, it yields that

$$[y, d(y)] = 0. \tag{7}$$

Now from (3), it is obvious to confirm that

$$[x, d(y) + y]rd(y) = 0 \quad \text{for all } x, r \in R. \tag{8}$$

Since R is a prime ring, we get $d(y) = 0$ or $[x, d(y) + y] = 0$. For the first case, the hypothesis (1) reduces to

$$[d(x), y] - [x, y] = 0 \quad \text{for all } x \in R. \tag{9}$$

Replacing x by xt in (9), we have

$$d(x)[t, y] + [t, y]d(t) = 0 \quad \text{for all } t, x \in R. \tag{10}$$

Putting tr instead of t in (10), we obviously get

$$d(x)t[r, y] + [x, y]td(r) = 0 \quad \text{for all } r, t, x \in R. \tag{11}$$

Taking $r = x$ in (11) gives

$$d(x)t[x, y] + [x, y]td(x) = 0 \quad \text{for all } t, x \in R. \tag{12}$$

By view of Lemma 1, we conclude that $d(x) = 0$ or $[x, y] = 0$.
Whence in any cases, it yields that

$$[x, y] = 0 \quad \text{for all } x \in R. \tag{13}$$

Therefore, it follows that $y \in Z(R)$.
Now if we consider

$$[x, d(y) + y] = 0 \quad \text{for all } x \in R. \tag{14}$$

Then $y + d(y) \in Z(R)$, which leads to

$$[d(x), y + d(y)] = 0 \quad \text{for all } x \in R. \tag{15}$$

From (1) together with (15), one can see that

$$[x, d(y)] - [x, y] = 0 \quad \text{for all } x \in R. \tag{16}$$

Adding (14) and (16), we can easily verify that

$$2[x, d(y)] = 0 \quad \text{for all } x \in R.$$

Using 2-torsion freeness, we obtain

$$[x, d(y)] = 0 \quad \text{for all } x \in R. \tag{17}$$

This fact gives $d(y) \in Z(R)$. Now it follows from (1) that

$$[d(x), y] - [x, y] = 0 \quad \text{for all } x \in R. \tag{18}$$

Using the same techniques as used above in (9), we get the required result.
Conversely, it is clear to see that if $y \in Z(R)$, then y is adjacent to x for all $x \in R$. ∎

The following corollary is an immediate consequence of the preceding Theorem.

Corollary 4. *Let R be a 2-torsion free prime ring and d a derivation of R. The following assertions are equivalent:*

(1) R is commutative integral domain.
(2) $\Gamma_d(R)$ is a complete graph.

Proof. For the nontrivial implication, assume that $\Gamma_d(R)$ is a complete graph. Then

$$[d(x), d(y)] + d[x, y] - [x, y] = 0 \quad \text{for all} \quad x, y \in R. \tag{19}$$

Using Theorem 1 we get $R = Z(R)$, thus R is a commutative integral domain. ∎

Our aim in the following proposition is to study the case of a star graph of a ring without unity.

Proposition 2. *Let R be a 2-torsion free prime ring and d a derivation of R. There is no derivation such that $\Gamma_d(R^*)$ is a star graph.*

Proof. Suppose that $\Gamma_d(R^*)$ is a star graph, there exists an only vertex $y \in R^*$ such that $[d(x), d(y)] + d[x, y] - [x, y] = 0$ for all $x \in R^*$, but none of the other vertices are adjacent to each other, thus by Theorem 1 we conclude that $y \in Z(R)^*$ and $|Z(R)| = 1$, also if $y \in Z(R)^*$ implies that $2y \in Z(R)^*$ we conclude that $y = 2y$, we find that $y = 0$, a contradiction. Consequently, $\Gamma_d(R^*)$ is not a star graph. ∎

Lemma 2. *Let R be a semi-prime ring, F an arbitrary map of R and H a generalized derivation of R associated with a derivation d such that*

$$[d(x), F(y)] = [H(y), x] \textit{ for all } x, y \in R$$

then one of the following assertions holds:
(1) $d = H = 0$;
(2) R contains a non-zero central ideal.

Proof. Assume that

$$[d(x), F(y)] = [H(y), x] \quad \text{for all} \quad x, y \in R. \tag{20}$$

If $F(y_0) \in Z(R)$, the center of R, for some $y_0 \in R$, then from Eq.(20) if follows that $[H(y_0), x] = 0$ for any $x \in R$. Thus $H(y_0) \in R$.
Let now consider $y_0 \in R$ be such that $b = F(y_0) \notin Z(R)$ and let $c = H(y_0)$.
From Eq.(20) we have

$$[d(x), b] = [c, x] \quad \text{for all} \quad x \in R. \tag{21}$$

If we denote by $\delta(x) = [x, b]$, the inner derivation induced by b, the relation (21) says that the composition δd is a derivation of R, more precisely it is the inner derivation induced by $-c$. By corollary 3 in [7], we have that $\delta d = 0$, which implies $[x, c] = 0$ for any $x \in R$, that is again $c = H(y_0) \in Z(R)$.
Therefore $H(y) \in Z(R)$, for any $y \in R$. If $0 \notin H(x_0) \in Z(R)$, for some $x_0 \in R$, then R contains a non zero central ideal generated by $H(x_0)$.
On the other hand, if $H(x) = 0$, for any $x \in R$, it follows

$$0 = H(xy) = H(x)y + xd(y) \quad \text{for all} \quad x, y \in R. \tag{22}$$

Implying that $Rd(R) = (0)$ and, by the semiprimeness of R, d=0. ∎

Proposition 3. *Let R be a semi-prime ring and d a non-zero derivation of R. If $\Gamma_d(R)$ is a complete graph then R contains a non-zero central ideal.*

Proof. Assume that $\Gamma_d(R)$ is a complete graph, then

$$[d(x), d(y)] + [d(x), y] + [x, d(y)] - [x, y] = 0 \quad \text{for all} \quad x, y \in R. \tag{23}$$

This can be rewritten as

$$[d(x), d(y) + y] + [x, d(y) - y] = 0 \quad \text{for all} \quad x, y \in R. \tag{24}$$

Putting $F(y) = d(y) + y$ as a generalized derivation of R associated with the derivation d, and the same for $H(y) = d(y) - y$, then the expression (24) becomes

$$[d(x), F(y)] = [H(y), x] \quad \text{for all} \quad x, y \in R. \tag{25}$$

Applying Lemma 2, it yields that R contains a non-zero central ideal. ∎

Let R_1, R_2 be two rings, d_1 and d_2 two derivations of R_1 and R_2 respectively. Setting $D(x, y) = (d_1(x), d_2(y))$ for all $(x, y) \in R_1 \times R_2$, obviously D is a derivation of $R_1 \times R_2$.

Theorem 2. *$\Gamma_{d_1}(R_1)$ and $\Gamma_{d_2}(R_2)$ are complete graphs if and only if $\Gamma_D(R_1 \times R_2)$ is a complete graph.*

Proof. Assume that $\Gamma_{d_1}(R_1)$ and $\Gamma_{d_2}(R_2)$ are complete graphs, if d_1, d_2 are derivations on R_1 and R_2 respectively then $D(x, x') = (d_1(x), d_2(x'))$ is a derivation on $R_1 \times R_2$. Now, we need to prove that

$$[D(x, x'), D(y, y')] + D[(x, x'), (y, y')] - [(x, x'), (y, y')] = (0, 0), \tag{26}$$

for all $(x, x'), (y, y') \in R_1 \times R_2$. Let's simplify the previous expression:

$[D(x, x'), D(y, y')] + D[(x, x'), (y, y')] - [(x, x'), (y, y')] =$
$[(d_1(x), d_2(x')), (d_1(y), d_2(y'))] + [(d_1(x), d_2(x')), (y, y')] + [(x, x'), (d_1(y), d_2(y'))] - [(x, x'), (y, y')] =$
$([d_1(x), d_1(y)] + d_1[x, y] - [x, y], [d_2(x), d_2(y)] + d_2[x, y] - [x, y])$.

Since $\Gamma_{d_1}(R_1)$ and $\Gamma_{d_2}(R_2)$ are complete graphs, we conclude that

$$[D(x, x'), D(y, y')] + D[(x, x'), (y, y')] - [(x, x'), (y, y')] = (0, 0), \tag{27}$$

for all $(x, x'), (y, y') \in R_1 \times R_2$. Which means that $\Gamma_D(R_1 \times R_2)$ is a complete graph.
Conversely, D is a derivation defined on $R_1 \times R_2$ such that
$D(x, x') = (d_1(x), d_2(x'))$ for all $(x, x') \in R_1 \times R_2$, then we can write

$$D((x, x')(y, y')) = (x, x')D(y, y') + D(x, x')(y, y') \quad \text{for all} \quad (x, x'), (y, y') \in R_1 \times R_2. \tag{28}$$

Replacing D by its expression, we find that

$$(d_1(xy), d_2(x'y')) = (x, x')(d_1(y), d_2(y')) + (d_1(x), d_2(x'))(y, y'), \tag{29}$$

for all $(x, x'), (y, y') \in R_1 \times R_2$. Developing the Eq.(29), we get

$$(d_1(xy), d_2(x'y')) = (xd_1(y) + d_1(x)y, x'd_2(y') + d_2(x')y'), \tag{30}$$

for all $(x, x'), (y, y') \in R_1 \times R_2$. Therefore, it follows that $d_1(xy) = xd_1(y) + d_1(x)y$ and $d_2(x'y') = x'd_2(y') + d_2(x')y'$, so d_1 and d_2 are derivations on R_1 and R_2 respectively.
Now assume that $\Gamma_D(R_1 \times R_2)$ is a complete graph, we have

$$[D(x, x'), D(y, y')] + D[(x, x'), (y, y')] - [(x, x'), (y, y')] = (0, 0), \tag{31}$$

for all $(x, x'), (y, y') \in R_1 \times R_2$. Using the same techniques as used above in (26), we get the required result. ∎

The primeness hypothesis in Theorem 1 is essential, as shown by the following example.

Example 1.
Let us consider the 2-torsion free ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}/3\mathbb{Z} \right\}$.
Define a derivation d of R by setting d $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Since $\Gamma_d(R)$ is as follows;

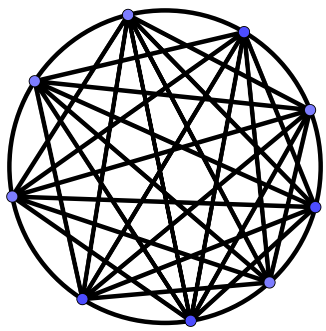


Figure 6: The graph $\Gamma_d(R)$.

as a result, $\Gamma_d(R)$ is a complete graph, however there is a vertex y adjacent to x for all $x \in R$ but $y \notin Z(R)$.

The following example proves that the condition "$char(R) \neq 2$" is necessary in Theorem 1.

Example 2.

Consider the prime ring $R = M_2(\mathbb{Z}/2\mathbb{Z})$. Let d be a derivation of R defined by

$d \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. Since the graph $\Gamma_d(R)$ is as follows;

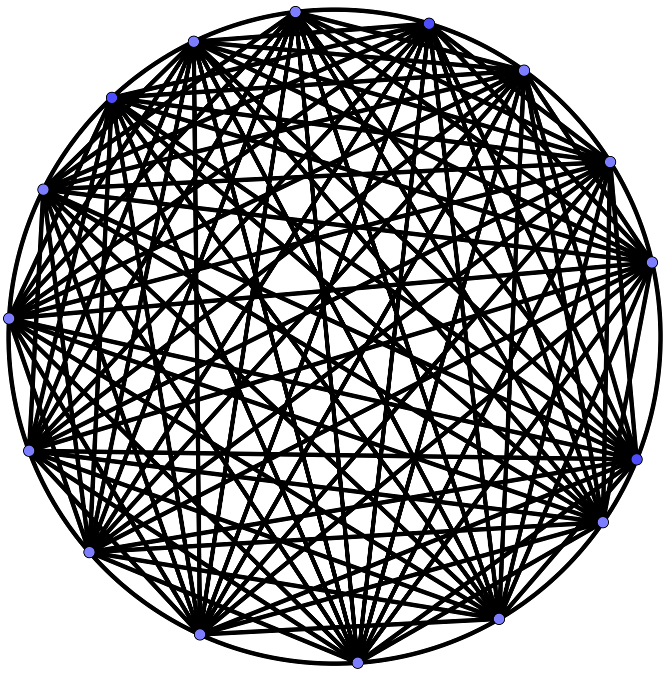


Figure 7: The graph $\Gamma_d(R)$.

then $\Gamma_d(R)$ is a complete graph, however there is a vertex y adjacent to x for all $x \in R$ but $y \notin Z(R)$.

The following example demonstrates that Theorem 1 cannot be extended to semiprime rings.

Example 3.

Let $R = M_2(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/3\mathbb{Z}$ and d the derivation defined on $M_2(\mathbb{Z}/2\mathbb{Z})$ by $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. It is straightforward to check that R is a non commutative semiprime ring. Moreover, the additive mapping $D : R \longrightarrow R$ defined by

$$D\big(M, N\big) = \big(d(M), 0\big)$$

is a derivation of R satisfying

$$[D(x), D(y)] + D[x, y] - [x, y] = 0 \text{ for all } x, y \in R.$$

Accordingly, $\Gamma_D(R)$ is a complete graph, however there is a vertex y adjacent to x for all $x \in R$ but $y \notin Z(R)$.

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